Lesson 13 (3.5)

Today: * Limits involving trig. functions
* Derivatives of trig. functions

Office Hours: MWF 2:45PM - 4:15PM

Announcements: * Feasting with faculty

Every Thursday: 11AM - 12:15PM
WINDSOR DINING HALL.

* Exam 2: Wednesday, Oct 15th 8PM - 9PM
* Final Exam: Monday, Dec 15th 1PM - 3PM
* HW 13,14: Due on Tuesday
* Quiz 8 (Lesson 12): On Tuesday.

Review: Rules of Differentiation

$$\frac{d}{dx}(c) = 0 \quad \text{for any constant } c$$

$$\frac{d}{dx}(x^n) = n \cdot x^{n-1} \quad \text{for any } n$$

$$\frac{d}{dx}(e^x) = e^x$$

$$\frac{d}{dx}\left[c \cdot f(x)\right] = c \cdot \frac{d}{dx} f(x)$$

$$\frac{d}{dx}\left[f(x) + g(x)\right] = \frac{d}{dx}\left[f(x)\right] + \frac{d}{dx}\left[g(x)\right]$$

$$\frac{d}{dx}\left[f(x) - g(x)\right] = \frac{d}{dx}\left[f(x)\right] - \frac{d}{dx}\left[g(x)\right]$$

$$\frac{d}{dx}\left[f(x)\,g(x)\right] = f'(x)\,g(x) + f(x)\,g'(x)$$

$$\frac{d}{dx}\left[\frac{f(x)}{g(x)}\right] = \frac{f'(x)\,g(x) - f(x)\,g'(x)}{(g(x))^2}$$

Today: Derivatives of Trig Functions

$$\frac{d}{dx}(\sin x) = \cos x$$

$$\frac{d}{dx}(\cos x) = -\sin x$$

$$\frac{d}{dx}(\tan x) = \sec^2 x$$

$$\frac{d}{dx}(\sec x) = \sec x \tan x$$

$$\frac{d}{dx}(\cot x) = -\operatorname{cosec}^2 x$$

$$\frac{d}{dx}(\operatorname{cosec} x) = -\operatorname{cosec} x \cot x$$

Use Defn:

$$f'(x) = \lim_{h \to 0} \frac{f(x+h) - f(x)}{h}$$

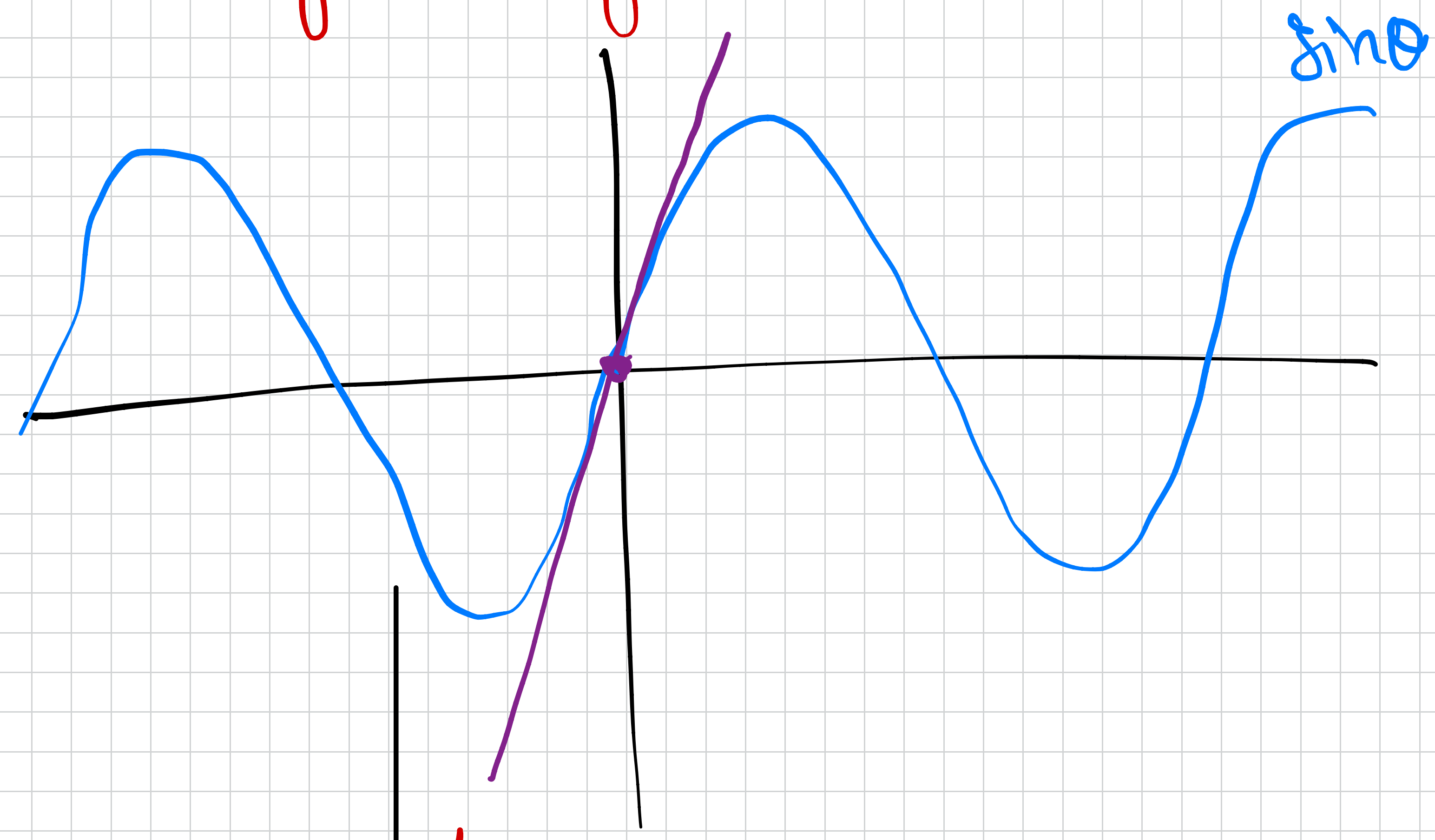
* use Trig identities
* Some limits involving sine & cosine

$$\frac{d}{dx}\left(\frac{\sin x}{\cos x}\right) \rightsquigarrow \text{Quotient Rule}$$

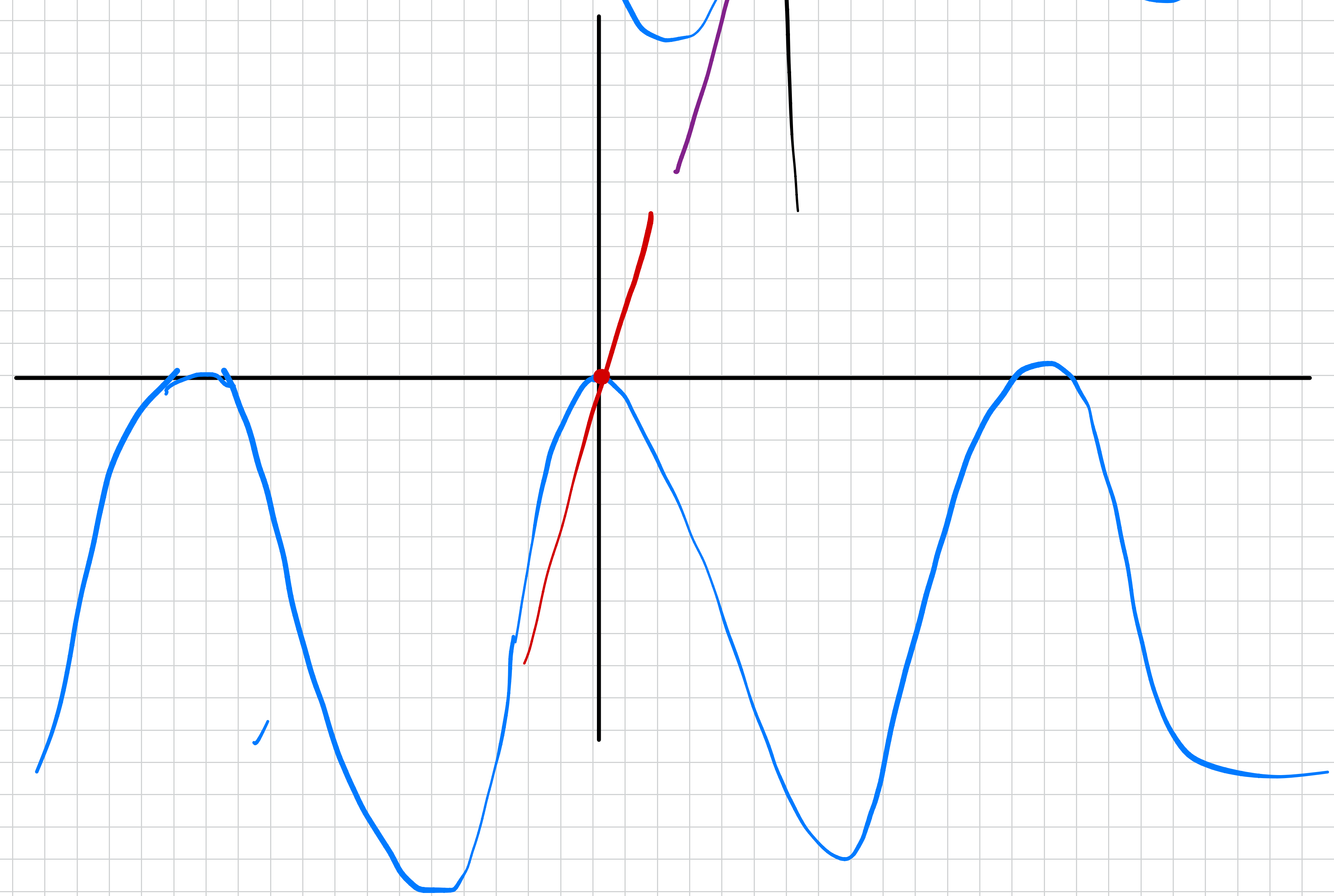
Important Limits involving Trig functions

① $$\lim_{\theta \to 0} \frac{\sin\theta}{\theta} = 1$$

also: $$\lim_{\theta \to 0} \frac{\theta}{\sin\theta} = 1$$



② $$\lim_{\theta \to 0} \frac{\cos\theta - 1}{\theta} = 0$$



eg: evaluate $\lim\limits_{x\to 0} \frac{\sin(2x)}{x}$

we know $\lim\limits_{\theta\to 0} \frac{\sin\theta}{\theta} = 1$, Let $\theta = 2x$

$x \to 0$; $\theta \to 0$

$$\lim_{\theta\to 0} \frac{\sin\theta}{(\theta/2)} = \lim_{\theta\to 0} 2 \cdot \frac{\sin\theta}{\theta} \to 1$$

$$= 2$$

Alternatively: $\lim\limits_{x\to 0} \frac{\sin 2x}{x} = \lim\limits_{x\to 0} 2 \cdot \frac{\sin 2x}{2x} \to 1 = 2$

eg: evaluate $\lim_{x \to 0} \frac{x}{\sin(3x)}$ $\left[\lim_{\theta \to 0} \frac{\theta}{\sin\theta} = 1 \right]$

$$= \lim_{x \to 0} \frac{1}{3} \cdot \underbrace{\frac{3x}{\sin(3x)}}_{\to 1}$$

$$= \frac{1}{3}$$

eg: Evaluate $\lim_{x\to 0} \frac{\sin(2x)}{\sin(3x)}$

just seen! $\lim_{x\to 0} \frac{\sin 2x}{x} = 2$, $\lim_{x\to 0} \frac{x}{\sin 3x} = \frac{1}{3}$

$$\lim_{x\to 0} \underbrace{\frac{\sin 2x}{x}}_{2} \cdot \underbrace{\frac{x}{\sin 3x}}_{1/3} = \boxed{\frac{2}{3}}$$

Generalize: for any $m, n \neq 0$

$$\lim_{x\to 0} \frac{\sin(mx)}{\sin(nx)} = \lim_{x\to 0} \underbrace{\frac{\sin(mx)}{x}}_{m} \cdot \underbrace{\frac{x}{\sin(nx)}}_{1/n} = \frac{m}{n}$$

Derivatives of Trigonometric functions

$$\frac{d}{dx}(\sin x) = \lim_{h\to 0} \frac{\boxed{\sin(x+h)} - \sin(x)}{h}$$

Trig identity! $\sin(A+B) = \sin A \cos B + \cos A \sin B$

$\sin(x+h) = \sin x \cosh + \cos x \sinh$

$$= \lim_{h\to 0} \frac{\boxed{\sin(x)\cosh} + \cos x \sinh - \boxed{\sin x}}{h}$$

$$= \lim_{h\to 0} \sin x \left[\frac{\cosh - 1}{h}\right] + \cos x \left[\frac{\sinh}{h}\right] = \cos x$$

? 0   ? 1

$$\boxed{\frac{d}{dx}\sin x = \cos x}$$

$$\frac{d}{dx}(\cos x) = \lim_{h \to 0} \frac{\cos(x+h) - \cos x}{h}$$

Trig identity:

$$\cos(A+B) = \cos A \cos B - \sin A \sin B$$

$$\cos(x+h) = \cos x \cosh - \sin x \sinh$$

$$= \lim_{h \to 0} \frac{\cos x \cosh - \sin x \sinh - \cos x}{h}$$

$$= \lim_{h \to 0} \cos x \left[\frac{\cosh - 1}{h}\right]^{0} - \sin x \left[\frac{\sinh}{h}\right]^{1} = -\sin x$$

$$\boxed{\frac{d}{dx} \cos x = -\sin x}$$

$$\frac{d}{dx}(\tan x)$$ ⇝ use: ① $\tan x = \frac{\sin x}{\cos x}$

② Quotient Rule

$$\boxed{\frac{d}{dx}\tan x = \sec^2 x}$$

$$= \frac{d}{dx}\left[\frac{\sin x}{\cos x}\right]$$

$$= \frac{\left(\frac{d}{dx}[\sin x]\right)\cos x - \sin x\left(\frac{d}{dx}[\cos x]\right)}{(\cos x)^2}$$

$\cos x$ $-\sin x$

$$= \frac{\cos^2 x + \sin^2 x}{\cos^2 x}$$

$= 1$ ⇝ Trig identity $\cos^2\theta + \sin^2\theta = 1$

$$= \frac{1}{\cos^2 x} = \sec^2 x$$

$$\frac{d}{dx}[\cot x] = \frac{d}{dt}\left[\frac{\cos x}{\sin x}\right]$$ ⇝ compute in a same way as $\frac{d}{dx}\tan x$ we compute! ⇝

$$\boxed{\frac{d}{dt}\cot x = -\operatorname{cosec}^2 x}$$

$\frac{d}{dx}(\sec x)$ ⇝ use ① $\sec x = \frac{1}{\cos x}$

② Quotient Rule

$$= \frac{d}{dx}\left[\frac{1}{\cos x}\right]$$

$$= \frac{\left(\frac{d}{dx}\cancel{1}\right)^{0}\cos x - 1\left(\cancel{\frac{d}{dx}\cos x}\right)^{-\sin x}}{(\cos x)^2}$$

$$= \frac{\sin x}{\cos^2 x} = \frac{\sin x}{\cos x}\cdot\frac{1}{\cos x}$$

$$= \tan x \sec x.$$

$$\boxed{\frac{d}{dx}\sec x = \tan x \sec x}$$

$$\frac{d}{dx}(\operatorname{cosec} x) = \frac{d}{dx}\left(\frac{1}{\sin x}\right)$$

compute similarly

$$\boxed{\frac{d}{dx}\operatorname{cosec} x = -\cot x \operatorname{cosec} x}$$

Summary: Derivatives of Trig Functions

$$\frac{d}{dx}(\sin x) = \cos x$$

$$\frac{d}{dx}(\cos x) = -\sin x$$

$$\frac{d}{dx}(\tan x) = \sec^2 x$$

$$\frac{d}{dx}(\sec x) = \sec x \tan x$$

$$\frac{d}{dx}(\cot x) = -\operatorname{cosec}^2 x$$

$$\frac{d}{dx}(\operatorname{cosec} x) = -\operatorname{cosec} x \cot x$$

eg:

$$\frac{d}{dx}\left[\frac{\tan x - 1}{\sec x}\right]$$

Quotient Rule

Product Rule $(\tan x - 1)\sec x$

$\tan x = \frac{\sin x}{\cos x}$, $\sec x = \frac{1}{\cos x}$

$$\frac{\tan x - 1}{\sec x} = \left(\frac{\sin x}{\cos x} - 1\right)\cos x$$

$$= \sin x - \cos x$$

$$\frac{d}{dx}\left[\frac{\tan x - 1}{\sec x}\right] = \frac{d}{dx}\left[\sin x - \cos x\right]$$

$$= \cos x + \sin x,$$

eg:

$$\frac{d}{dx}\left(\cos x \tan x - e^x \sec x\right).$$

$$\cos x \cdot \frac{\sin x}{\cos x} = \sin x$$

$$= \frac{d}{dx}\left[\sin x - e^x \sec x\right] = \cancel{\frac{d}{dx}\sin x} - \frac{d}{dx}\left[e^x \sec x\right]$$

$\cos x$

product Rule

$$\left(\frac{d}{dx}e^x\right)\sec x + e^x\left(\frac{d}{dx}\sec x\right)$$

$$e^x \sec x + e^x \sec x \tan x$$

$$= \cos x - \left(e^x \sec x + e^x \sec x \tan x\right).$$

$$= \cos x - e^x s$$